“ON THE RECURSIVE SEQUENCE

$$x_{n+1} = \frac{A}{x_n} + \frac{B}{x_{n-1}}$$ ”

Thomas KIVENTIDIS
Department of Mathematics , University of Thessaloniki
54124 Thessaloniki , Greece

ABSTRACT . Consider the difference equation

$$x_{n+1} = \frac{A}{x_n} + \frac{B}{x_{n-1}} \quad , \ A > 0 \ , B > 0 \ , \ n = 0,1,2,... \qquad (1)$$

with initial conditions $x_0 x_{-1} < 0$, and we find initial points x_{-1} and x_0 for which Eq.(1) in well defined for all $n \geq 0$ .If $\{x_n\}$ is such a solution then this is positive or negative after $n \geq n_0$, or alternative sign for $n = -1, 0, 1, 2, ...$If the solution finely maintains the sign then converges [4] . Eq.(1) has 2- periodic solution with two cycle $\sqrt{B-A}$, $-\sqrt{B-A}$, when $B \in (A, +\infty)$.

We consider the case

$$x_{n+1} = \frac{-1}{x_n} + \frac{B}{x_{n-1}} \quad , B > 0 \ , \ n = 0,1,2,... \qquad (2)$$

with initial conditions $x_{-1}x_0 < 0$ [2] .

We show that every solution of the equation (2) , with $x_{-1}x_0 < 0$ is such that $x_n x_{n-1} < 0$, $\forall n \geq 0$ and converges to the periodic two solution $\sqrt{B+1}$, $-\sqrt{B+1}$ (2_0) which is locally asymptotically stable .

Keywords: *Difference equation , initial conditions , forward and backward solution ,alternating solution , positive (negative) solution , periodic solution , asymptotically stable , unstable , first linear approximation .*

AMS 1990 Mathematics Subject Classification : 39A10 .

1. INTRODUCTION

We consider the difference equation

$$x_{n+1} = \frac{A}{x_n} + \frac{B}{x_{n-1}} \quad , \ A > 0 \ , \ B > 0 \ , \ n = 0,1,2,.. \qquad (1)$$

and we assume the initial conditions $x_{-1}x_0$ such that $x_{-1}x_0 < 0$. [2]

From the relation $x_{-1}x_0 < 0$ we have

$$x_{-1} < 0 \ , \ x_0 > 0 \ \text{ or } \ x_{-1} > 0 \ , \ x_0 < 0 \ ,$$

and

$$x_1 = \frac{A}{x_0} + \frac{B}{x_{-1}} \quad \Rightarrow \quad x_1 x_0 = A + \frac{Bx_0}{x_{-1}}$$

$$\Rightarrow \quad x_1 x_0 < A \text{ , since } x_{-1}x_0 < 0 .$$

From the relation

$$x_1 x_0 < A \quad \text{, if } \; Ax_{-1} + Bx_0 \neq 0 \;,$$

it follows that $\qquad 0 < x_1 x_0 < A \qquad$ (i_1)

or $\qquad x_1 x_0 < 0 < A . \qquad$ (ii_1)

For $n = 1$, $\; x_2 = \dfrac{A}{x_1} + \dfrac{B}{x_0} \quad \Rightarrow \quad x_2 x_1 = A + \dfrac{Bx_1}{x_0}$

we have the following two cases :

- if (i_1) is true then $x_2 x_1 > 0$,
- if (ii_1) is true then $x_2 x_1 < A$.

  If $x_2 x_1 < A$ is true then it follows $\;(\text{if } Ax_0 + Bx_1 \neq 0)$

$$0 < x_2 x_1 < A \qquad (i_2)$$

$$\text{or} \quad x_2 x_1 < 0 < A . \qquad (ii_2)$$

For $n = 2$, $\; x_3 = \dfrac{A}{x_2} + \dfrac{B}{x_1} \quad \Rightarrow \quad x_3 x_2 = A + B\dfrac{x_2}{x_1}$,

we have the following two cases :

- if (i_2) is true then $x_3 x_2 > 0$,
- if (ii_2) is true then $x_3 x_2 < A$.

  If $x_3 x_2 < A$ is true then it follows $\;(\text{if } Ax_1 + Bx_2 \neq 0)$

$$0 < x_3 x_2 < A \qquad (i_3)$$

$$\text{or} \quad x_3 x_2 < 0 < A . \qquad (ii_3)$$

We continue similarly for x_4 , x_5 ,...
We observe from the above that :

PROPOSITION : Consider the difference equation (1) with $x_{-1}x_0 < 0$, if existy $n_0 \geq 0$ such that $x_{n_0+1}x_{n_0} > 0$ then we shall have $x_{n+1}x_n > 0$, $\forall n \geq n_0$, otherwise we shall have $x_{n-1}x_n < 0$, $\forall n \geq 0$, if $x_n \neq 0$, $\forall n \geq 0$.

2. EXISTENCE OF SOLUTIONS

On note that we have

$$x_1 = \frac{A}{x_0} + \frac{B}{x_{-1}} \quad \Rightarrow \quad x_1 x_{-1} = \frac{x_{-1}}{x_0} A + B \quad \Rightarrow \quad x_1 x_{-1} < B \quad ,$$

since $x_{-1}x_0 < 0$ from hypothesis.

THEOREM 1 : If $x_0 > 0$, $x_{-1} < -\frac{B}{A}x_0 < 0$ (resp. if $x_0 < 0$, $x_{-1} > -\frac{B}{A}x_0 > 0$) then $x_1 > 0$ (resp $x_1 < 0$) and we have $x_{n+1}x_n > 0$,$\forall n \geq 0$.

Also , if $x_0 > 0$, $-\frac{B}{A}x_0 < x_{-1} < 0$ (resp. if $x_0 < 0$, $0 < x_{-1} < -\frac{B}{A}x_0$) then $x_1 < 0$ (resp. $x_1 > 0$) and we have the statements of the proposition (&1).

Proof

We have $x_1 = \frac{A}{x_0} + \frac{B}{x_{-1}} = \frac{Ax_{-1} + Bx_0}{x_0 x_{-1}} > 0$ (since $x_0 x_{-1} < 0$) and

$0 < x_1 x_0 = A + \frac{Bx_0}{x_{-1}} < A$.

From the proposition (&1) it follows that $x_{n+1}x_n > 0$, $\forall n \geq 0$.
Also , from the relation (we can see that $x_1 < 0$)

$$x_1 = \frac{A}{x_0} + \frac{B}{x_{-1}} \Rightarrow 0 < x_1 x_{-1} = \frac{Ax_{-1}}{x_0} + B < B$$

and $x_2 = \frac{A}{x_1} + \frac{B}{x_0}$, we have

- if $x_0 > -\frac{B}{A}x_1 > 0$ then $x_1 x_2 > 0$ and $x_{n+1}x_n > 0$, $\forall\, n \geq 1$,
- if $0 < x_0 < -\frac{B}{A}x_1$ then $x_1 x_2 < 0$ and produce the statements of the proposition (&1) .

Respectively , from the relation (we can see now that $x_1 > 0$)

$$x_1 = \frac{A}{x_0} + \frac{B}{x_{-1}} \Rightarrow 0 < x_1 x_{-1} = \frac{Ax_{-1}}{x_0} + B < B$$

and $x_2 = \frac{A}{x_1} + \frac{B}{x_0}$ we have

- if $x_0 < -\frac{B}{A}x_1 < 0$ then $x_1 x_2 > 0$ and $x_{n+1}x_n > 0$, $\forall\, n \geq 1$,
- if $-\frac{B}{A}x_1 < x_0 < 0$ then we have $x_1 x_2 < 0$ and produce the statements of the proposition (&1) .

We have $x_2 > 0$,(or $x_2 < 0$) since $x_1 x_0 < 0$ and $x_0 < -\frac{B}{A}x_1 < 0$ (or $-\frac{B}{A}x_1 < x_0 < 0$).

We observe that (in the case $x_0 < 0 , 0 < x_{-1} < -\frac{B}{A}x_0$)

$$0 < x_1 < \frac{B}{x_{-1}} \Rightarrow -\frac{B^2}{Ax_{-1}} < -\frac{B}{A}x_1 < 0$$

and if $x_{-1}x_0 \le -\dfrac{B^2}{A} < 0$ then $x_0 < -\dfrac{B}{A}x_1 < 0$

and , must be $x_0 < -\sqrt{B} < 0$ (since $0 < x_{-1} < -\dfrac{B}{A}x_0$).

On note that $-\dfrac{B}{A}\left(\dfrac{A}{x_0} + \dfrac{B}{x_{-1}}\right) < x_0 < 0 \Rightarrow -\dfrac{B}{x_0} - \dfrac{B^2}{Ax_{-1}} < x_0 < 0$

from which it follows

$$\frac{B^2}{Ax_{-1}} > -\frac{B}{x_0} > 0 \Rightarrow 0 < x_{-1} < -\frac{B}{A}x_0 .$$

Thus , the two relations can both hold , respectively.
Similarly , we prove the other case .

We shall also prove that $x_n x_{n+1} > 0$ $\forall n \ge n_0$ and $0 < x_n x_{n+1} < A$, $\forall n \ge n_0 + 1$, cannot both hold .

Indeed , if $x_n x_{n+1} > 0$, $\forall n \ge n_0$ then we distinguish the following two cases :

- if $x_n > 0$, $x_{n-1} > 0$, $\forall n-1 \ge n_0$ then

$$x_{n+1} = \frac{A}{x_n} + \frac{B}{x_{n-1}} > \frac{A}{x_n} \quad \Rightarrow \quad x_n x_{n+1} > A > 0 \quad ,$$

- if $x_n < 0$, $x_{n-1} < 0$, $\forall n\text{-}1 \ge n_0$ then

$$x_{n+1} = \frac{A}{x_n} + \frac{B}{x_{n-1}} \Rightarrow - x_{n+1} = \frac{A}{-x_n} + \frac{B}{-x_{n-1}} > \frac{A}{-x_n} \Rightarrow$$

$$(-x_{n+1})(-x_n) > A > 0 \Rightarrow x_n x_{n+1} > A > 0 .$$

Thus , if $x_n x_{n+1} > 0$, $\forall n \ge n_0$, then $x_n x_{n+1} > A > 0$, $\forall n \ge n_0 + 1$, holds.

Conclusions : From the conditions of the Theorem 1 the solution $\{x_n\}$ of the difference equation (1) is defined for all $n = 0,1,2,...$ and we have the statements :

- if n_0 exists such that $x_{n_0+1}x_{n_0} > 0$ then

$$x_{n+1}x_n > 0 \ , \ \forall n = n_0 \ , \ n_0+1 , ...$$

which means that the solution has constant sign (positive or negative)for $n \ge n_0$.

- otherwise we have $x_{n+1}x_n < 0$, $\forall n = 0,1,2,...$ which means that the solution has alternative sign for $n = -1,0,1,2,...$ (if $x_n \ne 0$, $\forall n$).

We then have the following statements :

α) if solution $\{x_n\}$of Eq.(1) has constant sign then (see[4]) it converges to number $\sqrt{A+B}$ (if positive) or to number $-\sqrt{A+B}$ (if negative) , where $A > 0$, $B > 0$.

β) observe that if $B > A > 0$ then Eq.(1) has the 2-periodic solution , with two cycle $\sqrt{B-A}$, $-\sqrt{B-A}$.

γ) one solution $\{x_n\}$ of Eq. (1) can have alternative sign if :

for $\quad -\dfrac{B}{A}x_1 < x_0 < 0 \ , \ 0 < x_{-1} < -\dfrac{B}{A}x_0 \quad$ (then $x_1 > 0$)

or for $\quad 0 < x_0 < -\dfrac{B}{A}x_1 \ , \ -\dfrac{B}{A}x_0 < x_{-1} < 0 \quad$ (then $x_1 < 0$) hold ,

and there is no n_0 such that $x_{n_0+1}x_{n_0} \geq 0$.

On note that if the above alternating solution $\{x_n\}$ of Eq.(1) converges , then it should converge to zero , which is absurd .

3. BACKWARD SOLUTIONS

We consider the difference equation

$$x_{n+1} = \frac{A}{x_n} + \frac{B}{x_{n-1}} \quad , \ A > 0 \ , \ B > 0 \ , \ n = 0, -1, -2, \ldots \qquad (1)'$$

We have the points

$$n = 0 : \quad x_1 = \frac{A}{x_0} + \frac{B}{x_{-1}}$$

$$n = -1 : \quad x_0 = \frac{A}{x_{-1}} + \frac{B}{x_{-2}} \quad \Rightarrow \quad x_0 x_{-2} = \frac{Ax_{-2}}{x_{-1}} + B$$

$$n = -2 : \quad x_{-1} = \frac{A}{x_{-2}} + \frac{B}{x_{-3}} \quad \Rightarrow \quad x_{-1}x_{-2} = A + \frac{Bx_{-2}}{x_{-3}}$$

....

- If $x_0 > 0$, $x_{-1} < 0$ (since $x_0x_{-1} < 0$) then $x_{-2} > 0$ and $0 < x_0x_{-2} < B$. Since $x_{-1}x_{-2} < 0$ it follows that $x_{-3} < 0$.

  From the relation $x_{-2}x_{-3} = A + \dfrac{Bx_{-3}}{x_{-4}}$,

since $x_{-2} > 0$, $x_{-3} < 0$, it follows $x_{-4} > 0$.

Hence we have the solution $\{x_n\}$, $n = 0,-1,-2,\ldots$ of Eq. (1)′ with alternative sign .

- If $x_0 < 0$, $x_{-1} > 0$ (since $x_0x_{-1} < 0$ by hypothesis) then

$$x_{-2} < 0 \ \text{ and } \ 0 < x_0x_{-2} < B .$$

Since $x_{-1}x_{-2} < 0$ it follows $x_{-3} > 0$.

Hence we have the solution $\{x_n\}$, $n = 0,-1,-2,\ldots,$ of Eq. (1)′ with alternative sign .

Evidently , this solution of Eq. (1)′ with alternative sign cannot converges .

When $B > A > 0$ then Eq. (1)′ has the 2-periodic solution with two cycle $\sqrt{B-A}$, $-\sqrt{B-A}$, $B \in (A, +\infty)$.

4. STABILITY

Consider the difference equation

$$x_{n+1} = \frac{-1}{x_n} + \frac{B}{x_{n-1}} \quad , \quad B > 0 \quad , \quad n = 0,1,2,.... \qquad (2)$$

We observe that , when $x_{-1}x_0 < 0$ then :

- $$x_1 = \frac{-1}{x_0} + \frac{B}{x_{-1}} \Rightarrow x_{-1}x_1 = \frac{-x_{-1}}{x_0} + B > 0$$

  which means that $x_{-1}x_1 > 0$,

- $$x_2 = \frac{-1}{x_1} + \frac{B}{x_0} \Rightarrow x_2 x_0 = \frac{-x_0}{x_1} + B > 0$$

  which means that $x_2 x_0 > 0$,

and we find , generally , that $x_n x_{n+1} < 0$, $n = 0,1,2,...$

It is easy to see that (2) has 2-periodic solution with two cycle $\sqrt{B+1}$, $-\sqrt{B+1}$, $B > 0$ (2_0) .

We set

$$f(x,y) = \frac{-1}{x} + \frac{B}{y} \quad ,$$

then

$$\frac{\partial f}{\partial x} = \frac{1}{x^2} \quad , \quad \frac{\partial f}{\partial y} = -\frac{B}{y^2} \ .$$

We have , from the linearized analysis of Eq. (2) in two cycle $\bar{x} = \sqrt{B+1}$, $\bar{y} = -\sqrt{B+1}$, that

$$y_{n+1} - \frac{\partial f}{\partial x}(\bar{x},\bar{y})y_n - \frac{\partial f}{\partial y}(\bar{x},\bar{y})y_{n-1} = 0 \quad n = 0,1,2,..$$

$$\Rightarrow y_{n+1} - \frac{1}{B+1}y_n + \frac{B}{B+1}y_{n-1} = 0 \quad , \quad n = 0,1,2,... \qquad (3)$$

where is here

$$p = -\frac{1}{B+1} \quad , \quad q = \frac{B}{B+1} \quad , \quad B > 0$$

Then the criterium of asymptotically stability of Eq. (3) ([1])

$$|p| < q + 1 < 2$$

hold .On note that 2-periodic solution of Eq.(2) to go over the zero solution of the first linear approximation (3) which is locally asymptotically stable .

We observe , also , that Eq.(2) has the equilibriums $\sqrt{B-1}$, $-\sqrt{B-1}$, where $B \in (1,+\infty)$.

The first linear approximation about the equilibriums is

$$y_{n+1} - \frac{1}{B-1} y_n + \frac{B}{B-1} y_{n-1} = 0 \quad , \; n = 0,1,2,... \qquad (3)'$$

and it is not hold the criterium of asympotically stability

$$|p| < q + 1 < 2 ,$$

where

$$p = -\frac{1}{B-1} \quad , \quad q = \frac{B}{B-1} \quad , \quad B \in (1 ,+\infty) .$$

On continue the study of stability of Eq. (3)′ .

The characteristic equation of Eq. (3)′ is

$$\lambda^2 - \frac{1}{B-1}\lambda + \frac{B}{B-1} = 0 \Rightarrow (B-1)\lambda^2 - \lambda + B = 0 \qquad (4)$$

and has the roots

$$\lambda_1 = \frac{1 + \sqrt{1-4B(B-1)}}{2(B-1)} \quad , \quad \lambda_2 = \frac{1 - \sqrt{1-4B(B-1)}}{2(B-1)} .$$

We consider the discrimination $\Delta(B) = -4B^2 + 4B + 1$, $B > 1$:

- if $\Delta(B) = 0 \Rightarrow B = \frac{1+\sqrt{2}}{2}$ and Eq. (4) has the double root

$$\lambda_1 = \lambda_2 = \frac{1}{\sqrt{2}-1} > 1$$

then instability .

- if $B \in \left(1 , \frac{1+\sqrt{2}}{3}\right)$, when $\Delta(B) > 0$ and $|\lambda_1| > 1$, $|\lambda_2| > 1$,

then instability .

- if $B \in \left(\frac{1+\sqrt{2}}{2} , +\infty\right)$ when $\Delta(B) < 0$ and the roots of Eq.(4)

are complex with modulus r :

— if $B = \frac{3}{2}$ when $r = \frac{1}{2(B-1)} = 1$, then stability,

— if $B > \frac{3}{2}$ or $B \in \left(\frac{1+\sqrt{2}}{2}, \frac{3}{2}\right)$ when $r^2 = \frac{B}{B-1} > 1$

then instability.

We have, summarily, the results :

- if $B \in \left(1, \frac{3}{2}\right)$ then instability of Eq. (3)′ and of Eq. (2) in equilibriums

- if $B = \frac{3}{2}$ then stability of Eq. (3)′, without conclusion for Eq. (2).

- if $B > \frac{3}{2}$ then instability of Eq. (3)′ and of Eq. (2) in equilibriums.

THEOREM 2: We assume that $\{x_n\}$ is a none equilibrium solution of Eq.(2), with $x_{-1}x_0 < 0$, which defined for all $n \geq 0$.
Then the solution $\{x_n\}$ converges to two cycle $\sqrt{B+1}$, $-\sqrt{B+1}$ (2) of 2-periodic solution of Eq. (2).

Proof

We have show in the introduction that, if for initial conditions x_{-1}, x_0 hold $x_{-1}x_0 < 0$ then $x_n x_{n+1} < 0$, $\forall n \geq 0$ for a solution $\{x_n\}$ of Eq.(2).
We suppose that $x_{2n} > 0$ and $x_{2n+1} < 0$, for $n = 0,1,2,\ldots$
For $N \geq 0$ let

$$m_N = min\{x_{2N-2}\ ,\ -x_{2N-1}\ ,\ x_{2N}\ ,\ -x_{2N+1}\},$$

$$M_N = max\{x_{2N-2}\ ,\ -x_{2N-1}\ ,\ x_{2N}\ ,\ -x_{2N+1}\}.$$

It follows from Eq.(2)

$$x_{2n+1} = \frac{-1}{x_{2n}} + \frac{B}{x_{2n-1}}, \quad n = 0,1,2,\ldots\ldots\ldots \tag{2.1}$$

$$x_{2n} = \frac{-1}{x_{2n-1}} + \frac{B}{x_{2n-2}}, \quad n = 0,1,2,\ldots\ldots\ldots \tag{2.2}$$

then

$$x_{2n+2} = \frac{-1}{x_{2n+1}} + \frac{B}{x_{2n}} = \frac{-1}{-\frac{1}{x_{2n}} + \frac{B}{x_{2n-1}}} + \frac{B}{\frac{-1}{x_{2n-1}} + \frac{B}{x_{2n-2}}}, \quad n = 1,2,\ldots$$

The function

$$f(x,y,z)=\frac{-1}{-\frac{1}{x}+\frac{B}{y}}+\frac{B}{-\frac{1}{y}+\frac{B}{z}}=\frac{1}{\frac{1}{x}+\frac{B}{-y}}+\frac{B}{\frac{1}{-y}+\frac{B}{z}}$$

is increasing in $x>0$, $y<0$, $z>0$.
Therefore, for $n=N$, we obtain

$$x_{2N+2}=\frac{1}{\frac{1}{x_{2N}}+\frac{B}{(-x_{2N-1})}}+\frac{B}{\frac{1}{(-x_{2N-1})}+\frac{B}{x_{2N-2}}}\leq\frac{1}{\frac{1}{M_N}+\frac{B}{M_N}}+\frac{B}{\frac{1}{M_N}+\frac{B}{M_N}}=M_N ,$$

$$x_{2N+3}=\frac{-1}{x_{2N+2}}+\frac{B}{x_{2N+1}}=-\left[\frac{1}{\frac{1}{(-x_{2N+1})}+\frac{B}{x_{2N}}}+\frac{B}{\frac{1}{x_{2N}}+\frac{B}{(-x_{2N-1})}}\right]\geq -M_N$$

$$x_{2N+4}=\frac{-1}{x_{2N+3}}+\frac{B}{x_{2N+2}}=\frac{1}{\frac{1}{x_{2N+2}}+\frac{B}{(-x_{2N+1})}}+\frac{B}{\frac{1}{(-x_{2N+1})}+\frac{B}{x_{2N}}}$$

$$\leq\frac{1}{\frac{1}{M_N}+\frac{B}{M_N}}+\frac{B}{\frac{1}{M_N}+\frac{B}{M_N}}=M_N$$

and by induction $x_{2n}\leq M_N$ for $n\geq N$ and $-x_{2n+1}\leq M_N$, for $n\geq N$.
We prove, similarly, that $x_{2n}\geq m_N$ for $n\geq N$ and $-x_{2n+1}\geq m_N$, for $n\geq N$.

We write the difference equation (2.1) in the form

$$-x_{2n+1}=\frac{1}{x_{2n}}+\frac{B}{-x_{2n-1}} \quad , n=0,1,2,... \qquad (2.1)'$$

and Eq. (2.2)

$$x_{2n}=\frac{1}{-x_{2n-1}}+\frac{B}{x_{2n-2}} \quad , \; n=0,1,2,... \qquad (2.2)'$$

We assume that $x_{2n}>0$ and $x_{2n+1}<0$, $n=0,1,2,...$ then the sequence $\{y_n\}$, where

$$y_n=x_{2k} \;, \text{ if } n=2k \text{ and } y_n=-x_{2k+1} \;, \text{ if } n=2k+1$$

is a positive sequence $y_n>0$, $\forall n\geq 0$ and bounded (since $m_N\leq y_n\leq M_N$, for $k\geq N$).

From Eqs. (2.1)′ and (2.2)′ implies that the sequence $\{y_n\}$ is a positive solution of difference equation

$$y_{n+1} = \frac{1}{y_n} + \frac{B}{y_{n-1}} \quad , \; B > 0 \; , \; n = 0,1,2,\ldots \qquad (2)'$$

It was shown in [4] (see Theorem) that every positive solution of the difference equation (2)′ converges to the positive equilibrium $\sqrt{B+1}$.

Thus , the subsequences of the positive solution $\{y_n\}$ of Eq. (2)′ $y_{2k} = x_{2k}$, $k = 0,1,2\ldots\ldots$ and $y_{2k+1} = -x_{2k+1}$, $k = 0,1,2,\ldots$ converges , also , the same limit $\sqrt{B+1}$.

Finally , we have show that

$$\lim_{k\to+\infty} x_{2k} = \sqrt{B+1} \qquad \text{and} \qquad \lim_{k\to+\infty} x_{2k+1} = -\sqrt{B+1} \; .$$

REFERENCES

1. V.L. KOCIC and G. LADAS , " *Global Behavior of Nonlinear Difference Equations of Higher Order with Applications* ". Kluwer , Dordrecht , 1993 .

2. R . DEVAULT , L. GALMINAS , E.J. JANOWSKI and G. LADAS , " On the Recursive Sequence $x_{n+1} = ((A/x_n)+(B/x_{n-1}))$ ", *Journal of Difference Equation and Applications* ,6 , pp .121 – 125 (2000) .

3. R . DEVAULT , G. LADAS and S.W. SCHULTZ ," On the recursive sequence $x_{n+1} = \frac{A}{x_n} + \frac{1}{x_{n-2}}$ " , *Proceeding of the AMS* , *Vol* .126 (1998) , pp 3257 – 3261 .

4.CH . PHILOS , I . K. PURNARAS and Y.G . SFICAS , " Glodal attractivity in a nonlinear difference equation " , *Applied Mathematics and Computers* 62 (1994) pp. 249 – 258 .